

On the Running of Gauge Couplings in String Theory

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Based on work in progress with **Steven A. Abel** and **Keith R. Dienes**,
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Motivations

Facts

- ❖ Closed string theories are **UV/IR mixing** theories;
- ❖ They may have **divergences**.

Questions:

- ❖ How do we **treat** these **divergences**?
- ❖ Is it sensible to introduce a **spacetime energy scale** and a **renormalisation procedure** which respects the **UV/IR mixed structure**?

Outline

- ❖ **One-loop corrected gauge couplings** in closed String Theory;
- ❖ How to deal with **divergences**;
- ❖ *A modular invariant* regulator;
- ❖ An example of how to **use it** .

One-loop (threshold) correction to gauge couplings in closed String Theory

$$\Delta_\alpha = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left\{ \underbrace{\frac{i}{\pi} \frac{1}{|\eta(\tau)|^4} \sum_{(a,b) \neq (1,1)} \partial_\tau \left(\frac{\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}}{\eta(\tau)} \right) \left(Q_\alpha^2 - \frac{k_\alpha}{4\pi\tau_2} \right)}_{\text{Spacetime part}} \underbrace{C \begin{bmatrix} a \\ b \end{bmatrix}}_{\text{Internal part}} - b_a \right\}$$

[V. Kaplunovsky, 1987]

Subtraction of the contribution of massless states

$$b_a = \text{Str} \left[Q_\alpha^2 \left(\frac{1}{12} - S^2 \right) \right] \Big|_{\text{massless}}$$

helicity

$$\eta(\tau) \equiv e^{\frac{\pi}{12}i\tau} \prod_{\ell=1}^{\infty} (1 - e^{2\pi\ell\tau})$$

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) \equiv \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\alpha/2)^2} e^{2\pi i(n+\alpha/2)(z+\beta/2)}$$

- ❖ k_α — Kac-Moody level at which the group factor G_α is realised;
- ❖ Q_α — Cartan charge.

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[V. Kaplunovsky, 1987]

$$b_a = \text{Str} \left[Q_\alpha^2 \left(\frac{1}{12} - S^2 \right) \right] \Big|_{\text{massless}}$$

This subtraction is needed to make the integral finite
BUT
it breaks worldsheet modular invariance

Many different regulators are possible

- ❖ A simple **subtraction** of the divergence [Kaplunovsky, 1987]

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau) \longrightarrow \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} (F(\tau) - c\tau_2), \quad F(\tau) \sim c\tau_2, \quad \tau_2 \gg 1$$

Breaks modular invariance!

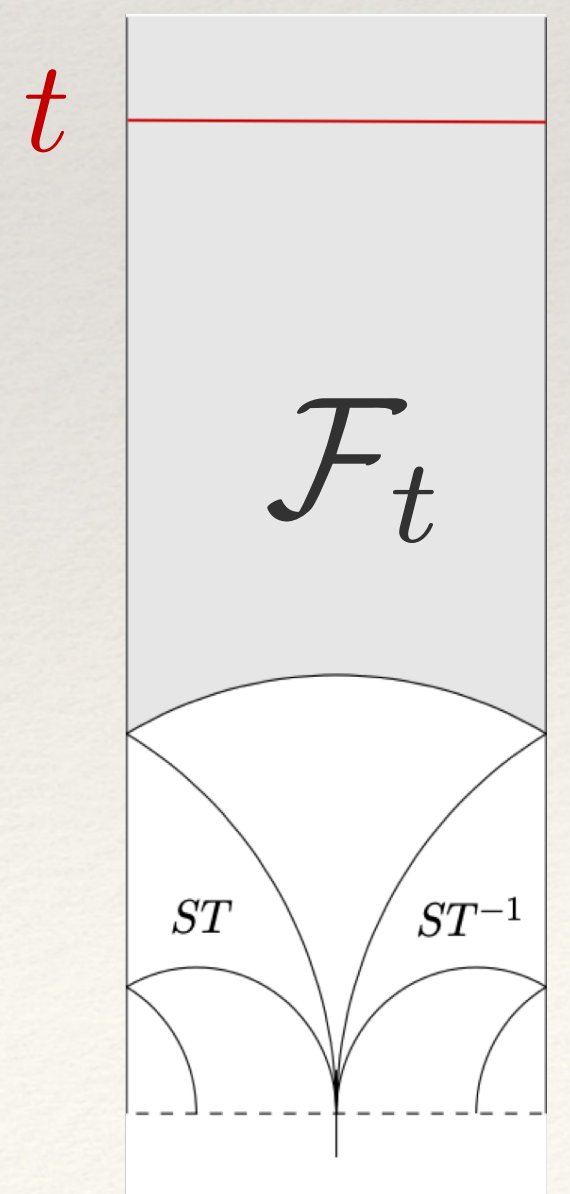
It corresponds to remove massless states => definition of threshold correction.

- ❖ Introducing a **cut-off** in the fundamental domain

$$\hat{I}(t) \equiv \int_{\mathcal{F}_t} \frac{d^2\tau}{\tau_2^2} F + \int_{\mathcal{F} - \mathcal{F}_t} \frac{d^2\tau}{\tau_2^2} (F - c\tau_2)$$

Breaks modular invariance!

[Don Zagier, 1981; C. Angelantonj, I. Florakis, B. Pioline, 2011, ArXiv: 1110.5318.]



Many different regulators are possible

- ❖ **Deform** the integrand in a modular invariant way

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F \longrightarrow \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau) \mathcal{G}(\tau)$$

The **regulator**:

- ❖ Should be **modular invariant**;
- ❖ Must **suppress any power-law divergence** for large τ_2 ;
- ❖ Should leave the rest of the theory intact as much as possible.

A modular invariant regulator



A possible choice

$$\mathcal{G}_\rho(a, \tau) \equiv \frac{1}{1 + \rho a^2} \frac{\rho}{\rho - 1} a^2 \frac{\partial}{\partial a} [Z_{\text{circ}}(\rho a, \tau) - Z_{\text{circ}}(a, \tau)]$$

$$\text{where } Z_{\text{circ}} \equiv \sqrt{\tau_2} \sum_{m, n \in \mathbb{Z}} e^{-\pi \tau_2 (m^2 a^2 + n^2 / a^2)} e^{2\pi i m n \tau_1}$$

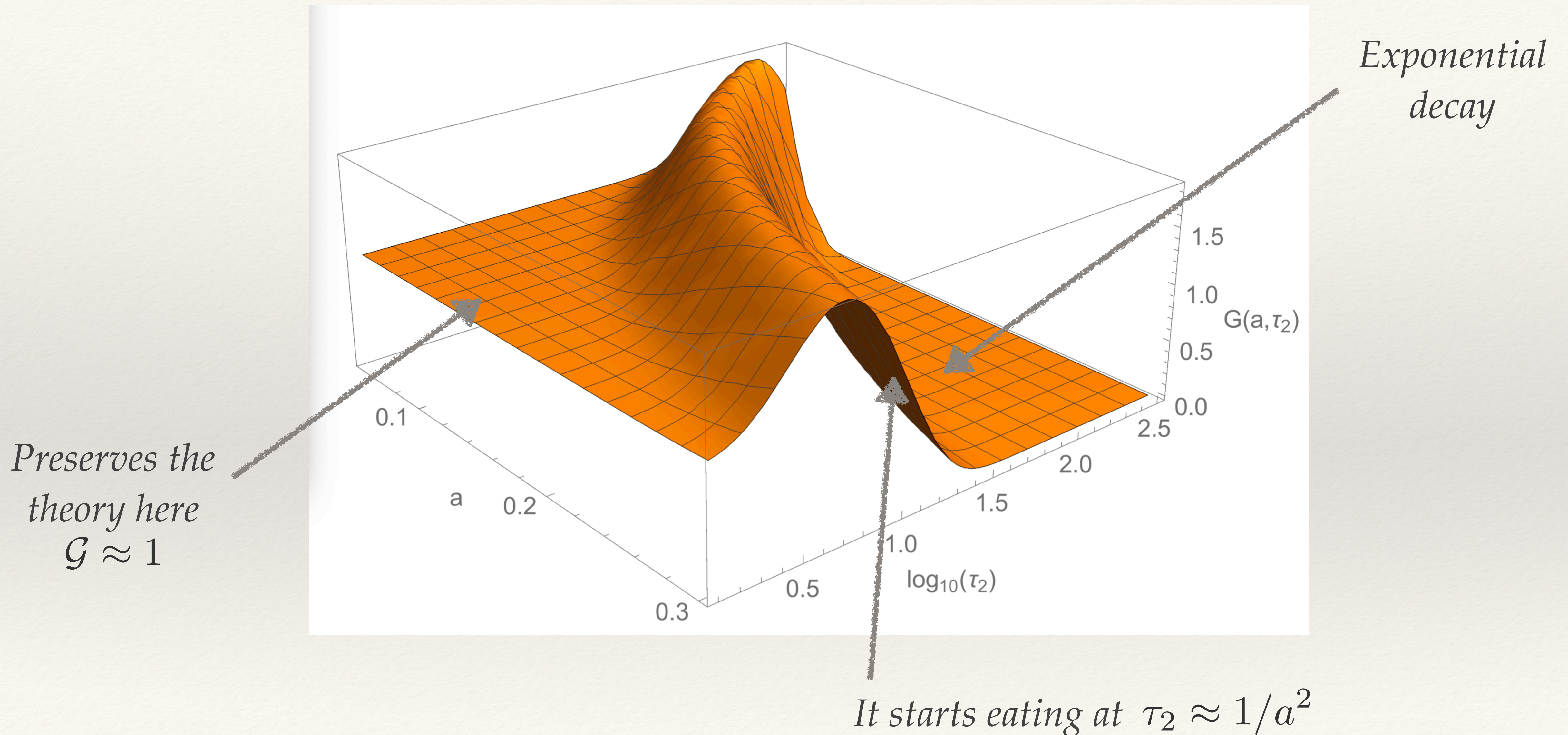
❖ Based upon Kiritsis and Kounnas regulator [E. Kiritsis, C. Kounnas, 1995] but **modified in a critical way**. Indeed:

❖ We can define $\mu^2 = \rho a^2 / \alpha'$ as an **energy scale**;

❖ **Modular invariance**  **UV/IR mixing**  **Scale duality symmetry**
 $\mu / M_s \longrightarrow M_s / \mu$

[S. A. Abel, K. R. Dienes, 2021, ArXiv: 2106:04622]

A modular invariant regulator



The case of 6D $\mathcal{N} = 1 \Rightarrow$ 4D $\mathcal{N} = 2$ toroidal compactifications

Famous result [L. Dixon, V. Kaplunovsky, J. Louis, 1990]

$$\Delta_a(T, U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} (\tau_2 \hat{U}_{2,2}(\tau) - \tau_2) = -\log \left(\frac{8\pi e^{1-\gamma}}{3\sqrt{3}} T_2 U_2 |\eta(T)|^4 |\eta(U)|^4 \right)$$

where:

$$\diamond \hat{U}_{2,2}(\tau) = \sum_{\vec{k}, \vec{\ell} \in \mathbb{Z}^2} e^{-\pi\tau_2 \alpha' M^2} e^{2\pi i \tau (k_2 \ell_1 - k_1 \ell_2)}$$

$$\diamond \alpha' M^2 = \frac{|k_1 + U k_2 + T \ell_1 + T U \ell_2|^2}{U_2 T_2}; (k_1, k_2, \ell_1, \ell_2) \in \mathbb{Z}^4$$

\diamond T and U are the so called **moduli of the (space-time) torus**.

The case of 6D $\mathcal{N} = 1 \Rightarrow$ 4D $\mathcal{N} = 2$ toroidal compactifications

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- ✓ Does not spoil the moduli dependence of the threshold;
- ✗ Worldsheet modular invariance is broken!

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$$\frac{1}{g^2} \Big|_{1\text{-loop}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2 \hat{U}_{2,2}(\tau) \mathcal{G}_\rho(a, \tau)$$



Everything is regulated and modular invariant!

Our result

$$\frac{1}{g(a, T, U)} \Big|_{1\text{-loop}} = -\frac{1}{1+2a^2} \left\{ \log(c T_2 U_2 |\eta(T)\eta(U)|^4) + 2 \log(\sqrt{2}a) + 4 \sum_{\gamma, \gamma' \in \Gamma_\infty \setminus \Gamma} \left[2\tilde{\mathcal{K}}_0^{(0,1)} \left(\frac{2\pi}{a\sqrt{\gamma \cdot T_2 \gamma' \cdot U_2}} \right) - \tilde{\mathcal{K}}_1^{(1,2)} \left(\frac{2\pi}{a\sqrt{\gamma \cdot T_2 \gamma' \cdot U_2}} \right) \right] \right\}$$

$$\diamond \quad \tilde{\mathcal{K}}_\nu^{(n,p)}(z, \rho) = \sum_{k,r=1}^{\infty} (krz)^n \left(K_\nu(krz/\rho) - \rho^p K_\nu(krz) \right)$$

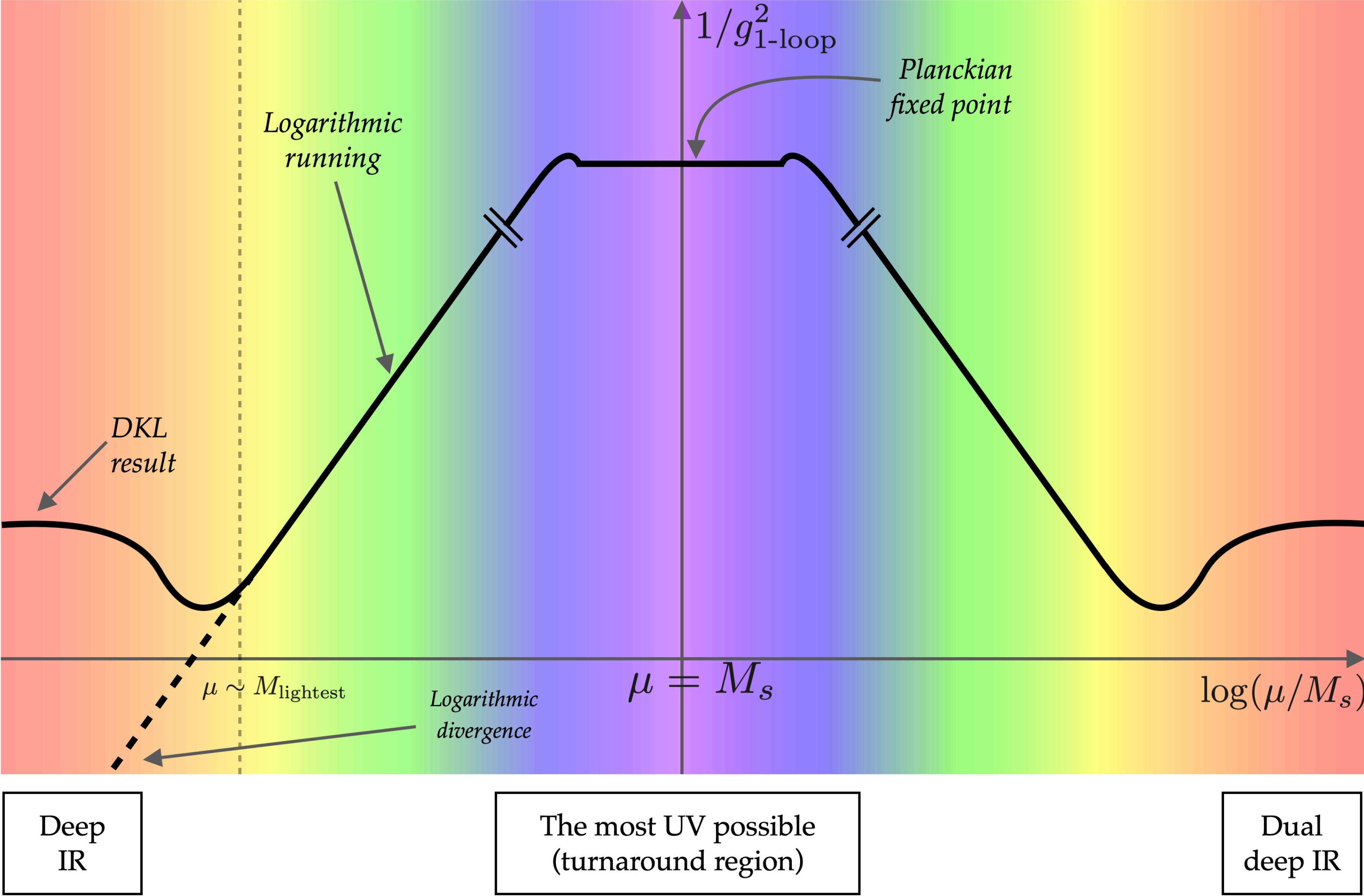
$$\diamond \quad \Gamma \equiv \text{SL}(2, \mathbb{Z}), \quad \Gamma_\infty \equiv \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad n \in \mathbb{Z}$$

$$\diamond \quad c \equiv 2\pi^2 e^{-2(\gamma+1)}$$

*Correction which restore
full worldsheet modular
invariance*

...contains all the required behaviour in limits...

Running of the gauge coupling



Conclusions and Outlook

- ❖ We **regulated** the theory fully respecting **modular invariance** and **UV/IR mixing properties**;
- ❖ We have seen how gauge couplings **run** in a particular class of compactifications;
- ❖ This formalism can be applied in **full generality to any model**. Indeed, using the methods developed in arXiv:2106.04622 for calculating the Higgs mass in string theory, we worked out the **general case** (to appear soon, arXiv: 2208:nnnnn)

Thanks for your attention!